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Spectral properties of disjointly strictly singular operators

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ABSTRACT

Spectral properties of strictly singular and disjointly strictly singular operators on Banach lattices are studied. We show that even in the case of positive operators, the whole spectral theory of strictly singular operators cannot be extended to disjointly strictly singular operators. However, several spectral properties of disjointly strictly singular operators are given.

1. Introduction

This note is devoted to the spectral theory of disjointly strictly singular and related classes of operators. In particular, it is well known that the spectra of positive operators on Banach lattices have richer structure than general operators. The monographs [1,2] are basic references for this theory.

Moreover, positive operators are a good source of models for applications in other mathematical disciplines, such as mathematical economy or biology (see [3]). In particular, in mathematical biology these operators have been useful for modeling the behavior of growing systems. In some of these applications, the interest focusses on finding positive equilibria in the evolution of a given system which turns out to be equivalent to finding positive eigenvectors for a positive operator (see [4]).

In particular, in [5], it was proved that for a certain class of mutation and selection regimes there exists a unique positive equilibrium density that is globally stable. This is achieved since the family of operators U_α, describing the behavior of the system under study, are dominated by an operator U with some compact power U^n:

$$0 \leq U_\alpha \leq U : L_1 \to L_1.$$

From this fact, by Aliprantis–Burkinshaw's domination theorem for positive compact operators [6], it follows that U_α^{3n} are compact operators too, and by Krein–Rutman's theorem there is a positive eigenvector f_α for each U_α so that (see [5, Theorem 4.1]):

$$U_\alpha(f_\alpha) = r(U_\alpha)f_\alpha.$$

It would be helpful to find out whether this technique can be extended to wider classes of operators (i.e. wider than the class of operators dominated by a positive compact operator).

As far as the spectral theory is concerned, compact operators have a very nice spectrum: it is an at most countable set whose only accumulation point can be 0, and every non-zero point in the spectrum is an eigenvalue whose corresponding eigenspace is finite dimensional. So our first interest will be to understand how the results for compact operators can be extended to operators with similar spectra.

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Moreover, we will study the spectral properties of disjointly strictly singular operators, which are a natural extension of strictly singular operators on Banach lattices [7]. In particular, we will show that even in the case of positive operators there exist disjointly strictly singular operators which are not Riesz. However, a stability property for the eigenvalues of disjointly strictly singular operators is given (see Theorems 4 and 5).

We will also introduce a related class of operators: Complementedly strictly singular operators. This class coincides with disjointly strictly singular operators in some spaces and carries some special structure regarding its spectrum (Corollaries 1 and 2).

The paper is organized as follows. In the next section we present basic facts concerning Riesz operators and Krein–Rutman's theorem, and in particular, we show that the arguments used above in Bürger's application work for positive operators dominated by strictly singular operators. In Section 3 the spectral properties of disjointly strictly singular operators are studied. Finally, Section 4 is devoted to the basic properties of the class of complementedly strictly singular operators, as well as its relation with strictly singular and disjointly strictly singular operators.

We refer the reader to [8,9,1,2] for any unexplained terminology concerning Banach lattices and positive operators.

2. Riesz operators and Krein–Rutman's theorem

As usual, given a Banach space X, $\mathcal{L}(X)$ (respectively $\mathcal{K}(X)$) denotes the space of bounded linear (resp. compact) operators $T : X \to X$.

Recall that an operator $T \in \mathcal{L}(X)$ is called Riesz when every $\lambda \in \sigma(T) \setminus \{0\}$ is an isolated point in $\sigma(T)$ and the corresponding spectral projection $P_\lambda(T)$ has finite rank [8, Section 7.5]. Equivalently, T is a Riesz operator if and only if its essential spectral radius, which is the spectral radius of the operator in the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$, is zero. Notice that if T is Riesz, then its spectrum is at most countable, 0 is the only possible point in the accumulation of $\sigma(T)$, and every $\lambda \in \sigma(T) \setminus \{0\}$ is an eigenvalue which is also a pole of the resolvent $R(\lambda, T)$.

It is clear that every compact operator is a Riesz operator, but this is a much larger class. Recall that an operator is strictly singular (or Kato) if it is never invertible when restricted to infinite dimensional subspaces. This class was introduced in [10] in connection with the perturbation of Fredholm operators. In particular, it holds that if S is strictly singular and F is a Fredholm operator (of index $h(F)$) then $S + F$ is also Fredholm (with $h(S + F) = h(F)$). In particular, this implies that every strictly singular operator is Riesz (cf. [8]).

Strictly singular operators, due to their infinite dimensional character, provide moreover an important tool for understanding the geometry of Banach spaces. There is a vast literature exploiting their properties and several new results related to them (see for instance [11,12]).

In connection with the application mentioned in the introduction, it would be helpful to know if a positive operator dominated by a Riesz operator is also Riesz. As far as we know, this problem remains open (see [13] for a discussion on this and other domination problems). However, a domination result holds for positive strictly singular operators [14]:

Theorem 1. *Let E be a Banach lattice, and $0 \le S \le T : E \to E$. If T is strictly singular, then S^4 is also strictly singular.*

Another important ingredient in Bürger's application is Krein–Rutman's theorem. It is well-known that the spectral radius of a positive operator is always a point of the spectrum [1, Proposition 4.1.1]. Krein–Rutman's theorem claims that for a *positive compact* operator T with non-zero spectral radius $r(T)$, $r(T)$ is an eigenvalue with a positive eigenvector. The proof of this fact can be extended to positive Riesz operators as follows (cf. [1, Theorem 4.1.4]):

Theorem 2. *Let $T : E \to E$ be a positive Riesz operator such that $r(T) > 0$. Then there exists a positive $x > 0$ in E such that $T(x) = r(T)x$.*

Proof. Since T is positive and $r(T) > 0$, by [1, Proposition 4.1.1] we have that $r(T) \in \sigma(T)$, and the resolvent $R(\lambda, T)$ is a positive operator in E for every $\lambda > r(T)$.

Since T is Riesz, it follows that $r(T)$ is a pole of the resolvent $R(\lambda, T)$. Let k be the order of this pole. Hence,

$$\lim_{\lambda \to r(T)} (\lambda - r(T))^k R(\lambda, T) \neq 0$$

so, if the limit is taken considering $\lambda > r(T)$, then for $x \in E$

$$S(x) = \lim_{\lambda \to r(T)^+} (\lambda - r(T))^k R(\lambda, T)(x)$$

defines a positive operator which is not identically zero. Let $x \in E_+$ be such that $S(x) \neq 0$. Since,

$$(r(T)I - T)S = \lim_{\lambda \to r(T)} (\lambda - r(T))^k (\lambda I - T) R(\lambda, T) = 0$$

it follows that $r(T)Sx = TSx$, so Sx is a positive eigenvector for T. □

We will see now that Krein–Rutman's theorem holds for positive operators dominated by strictly singular operators. It seams natural to ask whether using these results, similar results to that of [5] can be given, under weaker assumptions (that allow strictly singular operators into the picture).

Theorem 3. *Let E be a Banach lattice and let $0 \le S \le T : E \to E$ be positive operators with T strictly singular. If S has non-zero spectral radius $r(S) > 0$, then S has a positive eigenvector $x > 0$ such that $S(x) = r(S)x$.*

Proof. Since T is strictly singular, by Theorem 1, we have that S^4 is also strictly singular. In particular, S^4 is a Riesz operator, but this implies that S is also Riesz. Theorem 2 yields the result. □

Under quite general assumptions, strictly singular operators can be described in terms of two related notions: AM-compactness and disjoint strict singularity. Recall that an operator $T : E \to Y$ from a Banach lattice E to a Banach space Y is called

- *AM-compact* if it maps order bounded sets into compact sets,
- *disjointly strictly singular* if it is not invertible on the span of any disjoint sequence in E.

A recent result, [15, Theorem 2.4], asserts that for a Banach lattice E with finite cotype, an operator $T : E \to Y$ which is AM-compact and disjointly strictly singular is strictly singular (see also [16]).

Observe that the spectral theory of AM-compact operators is not as satisfactory as that of strictly singular operators. First, notice that every operator $T : \ell_2 \to \ell_2$ is AM-compact [1, p. 218], so any compact set $K \in \mathbb{C}$ can be the spectrum of an AM-compact operator. Moreover, the shift operator mapping each sequence $(x_1, x_2, \ldots)$ in ℓ_2 to $(0, x_1, x_2, \ldots)$, defines a positive operator $S : \ell_2 \to \ell_2$ with $r(S) = 1$ (since $\|S^n\| = 1$ for every $n \in \mathbb{N}$), but clearly $S(x) = x$ holds only when $x = 0$. This shows that Krein–Rutman's theorem does not hold for AM-compact operators.

According to the previous mentioned result from [15], since strictly singular operators have nice spectral properties and AM-compact operators are as bad as they can be, one might expect that disjointly strictly singular operators have better spectral properties. The following section is devoted to this discussion.

3. Spectra of disjointly strictly singular operators

We will focus now on spectral properties of disjointly strictly singular operators. Recall that an operator $T : E \to X$ between a Banach lattice E and a Banach space X is disjointly strictly singular (DSS) if it is never invertible when restricted to the span of a disjoint sequence. This class of operators contains that of strictly singular operators and have proved useful for understanding the properties of strictly singular operators on Banach lattices (see [15,14,17]).

Remark 1. Notice that on an infinite dimensional Banach lattice E, every DSS operator $T \in \mathcal{L}(E)$ satisfies $0 \in \sigma(T)$.

On certain spaces, the class of disjointly strictly singular operators coincides with that of strictly singular operators. This is the case for instance in atomic Banach lattices, $C(K)$ spaces [18] and $L_1(\mu)$ spaces:

Proposition 1. *Every DSS operator $T : L_1(\mu) \to L_1(\mu)$ is strictly singular.*

Proof. Let us see that if $T : L_1 \to L_1$ is DSS, then it is also ℓ_2-singular (i.e. T is not invertible on any subspace isomorphic to ℓ_2). Indeed, if this were not the case, then by Bourgain [19], T would be invertible when restricted to some subspace of L_1 of the form $(\bigoplus \ell_2)_1$, which actually consists of disjoint copies of subspaces isomorphic to ℓ_2; thus, T would be invertible on the span of a disjoint copy of ℓ_1. This contradiction shows that T is ℓ_2-singular, and by Flores et al. [15, Theorem A], T must be strictly singular. □

Similarly, we have the following.

Proposition 2. *Let E be a p-concave Banach lattice ($p < \infty$), and $T : E \to E$ a regular disjointly strictly singular operator. If $T : L_1 \to L_1$ is also disjointly strictly singular, then $T^2 : E \to E$ is strictly singular.*

Proof. First notice that since $T : E \to E$ is regular, by a change of density we can assume that $T : L_1 \to L_1$ is also bounded [20]. If $T^2 : E \to E$ is not strictly singular, then by Flores et al. [15], there must exist a sequence $(f_n)_n$ in E, equivalent to the unit vector basis of ℓ_2, and such that T^2 is an isomorphism when restricted to $[f_n]$. Since $T : E \to E$ is disjointly strictly singular, it follows that in both subspaces $[f_n]$ and $[Tf_n]$, the norms $\|\cdot\|_E$ and $\|\cdot\|_{L_1}$ are equivalent [21]. Thus, the extension $\widetilde{T} : L_1 \to L_1$ preserves an isomorphic copy of ℓ_2.

As above, by Bourgain's characterization of Dunford–Pettis operators on L_1 [19], it follows that $\widetilde{T}$ preserves a copy of $(\bigoplus \ell_2)_{\ell_1}$. However, this is a contradiction with the fact that $\widetilde{T}$ is disjointly strictly singular. □

In the following results, L_p will denote the space $L_p[0, 1]$ endowed with Lebesgue measure; however everything works for an $L_p(\mu)$ space over any finite measure.

For the eigenvalues of a DSS operator on L_p we have the following stability property.

Theorem 4. *Let $1 < p < 2$ and $T : L_p \to L_p$ be a DSS operator. The set of eigenvalues of $T : L_r \to L_r$ for any $r \in [p, 2)$ (and their corresponding eigenspaces) is independent of r.*

Proof. First, by Johnson and Jones [22] there is an isometry $J : L_p \to L_p$ such that $JTJ^{-1} : L_2 \to L_2$ is bounded. Clearly, since the eigenvalues of T coincide with those of JTJ^{-1}, we can suppose that T is bounded also on L_2. Moreover, by interpolation $T : L_r \to L_r$ is also bounded for any $r \in [p, 2]$.

Now, for $p < r < 2$ we clearly have $L_r \subset L_p$. Thus, every eigenvalue (respectively eigenvector) of $T : L_r \to L_r$ is an eigenvalue (resp. eigenvector) of $T : L_p \to L_p$. To see the converse, let λ be an eigenvalue of $T : L_p \to L_p$ and denote

$$X_\lambda = \ker(\lambda I - T) \subseteq L_p.$$

By Johnson and Schechtman [23, Proposition 1], X_λ embeds in L_r, so λ is also an eigenvalue (with the same eigenspace) for $T : L_r \to L_r$. $\square$

In the case of positive DSS operators the previous stability property can be further extended. Before giving this result we need several facts. Recall that an operator $T : E \to Y$ between a Banach lattice E and a Banach space Y is called M-weakly compact if $\|Tu_n\| \to 0$ for every disjoint normalized sequence (u_n) in E. Also recall that an operator $T : X \to E$ is called L-weakly compact if $\|y_n\| \to 0$ for every disjoint sequence in the solid hull of $T(B_X)$.

Lemma 1. *Let $T : L_p \to L_p$ be a positive operator $1 < p < \infty$. The following are equivalent:*

1. *T is disjointly strictly singular.*
2. *T is M-weakly compact.*
3. *T is L-weakly compact.*

Proof. Clearly, every M-weakly compact operator is disjointly strictly singular. For the converse, assume T is not M-weakly compact, so there is a disjoint normalized sequence (u_n) in L_p such that $\|Tu_n\| \geq \alpha > 0$. Observe that $(|u_n|)$ is also a disjoint normalized sequence, and so it is equivalent to the unit vector basis of ℓ_p. In particular $(|u_n|)$ as well as $(T|u_n|)$ are weakly null sequences of positive elements. It follows that $\|T|u_n|\|_{L_1} \to 0$, so by Kadeč and Pełczyński [21], $(T|u_n|)$ must be equivalent to a disjoint sequence in L_p. Therefore, the restriction $T|_{[|u_n|]}$ is an isomorphism, so T is not DSS. This proves the equivalence of the first two statements. The remaining equivalence follows from [1, Theorem 3.6.17]. $\square$

Notice that with exactly the same proof this fact also holds for reflexive disjointly homogeneous Banach lattices (see [16]).

The following result is an interpolation fact that may be interesting in its own.

Proposition 3. *Let $T : L_p \to L_p$ be a positive DSS operator for some $1 < p < \infty$. Then $T : L_r \to L_r$ is also DSS for every $1 < r < \infty$.*

Proof. First notice that by Weis [20], there is a positive isometry J on L_p such that $JTJ^{-1} : L_r \to L_r$ is bounded for any $1 \leq r \leq \infty$. Since the statement for T and JTJ^{-1} are equivalent, without loss of generality we replace T with JTJ^{-1}.

Given any set $A \subset [0, 1]$ of positive measure, let us define the operator $P_A(x) = \chi_A \cdot x$ which is bounded on L_q for every $1 \leq q \leq \infty$ with $\|P_A\|_{L_q} = 1$.

Suppose first $p \geq 2$. According to [24, Proposition 4.1], we have that for any sequence (A_n) of disjoint measurable sets in $[0, 1]$, $\lim_n \|TP_{A_n}\|_{L_p} = 0$. We claim that for any $1 < r < \infty$ it also holds that $\lim_n \|TP_{A_n}\|_{L_r} = 0$. Indeed, for $1 < r < p$, let $\frac{1}{r} = \theta + \frac{1-\theta}{p}$ with $\theta \in (0, 1)$. For any sequence (A_n) of disjoint measurable sets, by the Riesz interpolation theorem, we have

$$\|TP_{A_n}\|_{L_r} \leq \|TP_{A_n}\|_{L_1}^{\theta} \|TP_{A_n}\|_{L_p}^{(1-\theta)} \to 0$$

since $\|TP_{A_n}\|_{L_1}$ is bounded. A similar argument works for $p < r < \infty$ using that $\|TP_{A_n}\|_{L_\infty}$ is bounded.

Now, suppose that $T : L_r \to L_r$ is not DSS, by Lemma 1, this means that for some disjoint sequence (x_n) with $\|x_n\|_r = 1$ we have $\|Tx_n\|_r \geq \alpha > 0$. Let A_n denote the support of the element x_n. Hence,

$$\|TP_{A_n}\|_{L_r} \geq \frac{\|TP_{A_n}x_n\|_r}{\|x_n\|_r} = \|Tx_n\|_r \geq \alpha,$$

which contradicts the fact proved above that $\lim_n \|TP_{A_n}\|_{L_r} = 0$. Thus, $T : L_r \to L_r$ must be DSS.

It remains to prove the case when $p < 2$. Again, using [24, Proposition 4.1], it holds that $\lim_n \|P_{A_n}T\|_{L_p} = 0$ for any sequence (A_n) of disjoint measurable sets. Arguing as above, we can prove that in this case $\lim_n \|P_{A_n}T\|_{L_r} = 0$ also holds for any sequence of disjoint sets (A_n) and any $1 < r < \infty$.

Now, if $T : L_r \to L_r$ is not DSS, then Lemma 1 implies that there exists a disjoint sequence (y_n) in L_r with $|y_n| \leq |Tx_n|$ for some $\|x_n\|_r \leq 1$ and such that $\|y_n\|_r \geq \beta > 0$. Let A_n denote the support of the element y_n. Hence,

$$\|P_{A_n}T\|_{L_r} \geq \frac{\|P_{A_n}Tx_n\|_r}{\|x_n\|_r} \geq \|P_{A_n}Tx_n\|_r \geq \|y_n\|_r \geq \beta,$$

which contradicts the fact proved above that $\lim_n \|P_{A_n}T\|_{L_r} = 0$. Therefore, $T : L_r \to L_r$ must be DSS. $\square$

Now, we can finally prove the stability result for eigenvalues of positive DSS operators.

Theorem 5. *Let $T : L_p \to L_p$ be a positive DSS operator. The set of eigenvalues of $T : L_r \to L_r$ (and corresponding eigenspaces) for $1 < r < \infty$ is independent of r.*

Proof. Notice that without loss of generality we can assume that $T : L_r \to L_r$ is bounded with $\|T\|_{L_r} \le 1$ for every $1 \le r \le \infty$ [20]. Moreover, by Proposition 3, $T : L_r \to L_r$ is DSS for every $1 < r < \infty$.

Let $1 < r < q < \infty$. Since $L_q \subset L_r$ we have that any eigenvector for $T : L_q \to L_q$ is also an eigenvector for $T : L_r \to L_r$ associated to the same eigenvalue. To prove the converse we will follow the lines of [23, Proposition 1].

Let λ be an eigenvalue of $T : L_r \to L_r$ and consider

$$X_\lambda = \ker(\lambda I - T) \subset L_r.$$

We will see that X_λ also embeds in L_q. First, since $T : L_r \to L_r$ is DSS, by Lemma 1, we have that $T(B_{L_r})$ is an L-weakly compact set of L_r (see [1, Section 3.6]). Hence, by [1, Proposition 3.6.2], for every $\varepsilon > 0$ there is x_ε in L_r such that

$$T(B_{L_r}) \subset [-x_\varepsilon, x_\varepsilon] + \varepsilon B_{L_r}.$$

Now, if we truncate x_ε with some $M_\varepsilon > 0$ such that

$$\left(\int_{|x_\varepsilon|>M_\varepsilon} |x_\varepsilon|^r d\mu\right)^{\frac{1}{r}} \le \varepsilon,$$

we then have that

$$T(B_{L_r}) \subset M_\varepsilon B_{L_\infty} + 2\varepsilon B_{L_r}.$$

Now, since $Tx = \lambda x$ for $x \in X_\lambda$, for each $n \in \mathbb{N}$ we have

$$\lambda^n B_{X_\lambda} \subset T^n(B_{L_r}) \subset 2M_\varepsilon B_{L_\infty} + (2\varepsilon)^n B_{L_r}.$$

Therefore, for any unit vector $x \in X_\lambda$ we can write $x = x_n + y_n$ with $\|x_n\|_\infty \le 2M_\varepsilon \frac{1}{|\lambda|^n}$ and $\|y_n\|_r \le \left(\frac{2\varepsilon}{|\lambda|}\right)^n$. Hence, for every $n \in \mathbb{N}$ we have $x_{n+1} - x_n = y_n - y_{n+1}$ which satisfy

$$\|x_{n+1} - x_n\|_\infty \le 4M_\varepsilon \frac{1}{|\lambda|^{n+1}}, \quad \|y_n - y_{n+1}\|_r \le 2\left(\frac{2\varepsilon}{|\lambda|}\right)^n$$

as long as $\varepsilon \le \frac{|\lambda|}{2}$.

Since $r < q$, for $\theta = \frac{r}{q}$ we have

$$\|x_{n+1} - x_n\|_q \le \|x_{n+1} - x_n\|_\infty^{1-\theta} \|y_n - y_{n+1}\|_r^\theta \le 2\left(\frac{2M_\varepsilon}{|\lambda|}\right)^{1-\theta} \left[\frac{(2\varepsilon)^\theta}{|\lambda|}\right]^n$$

which is a summable sequence if $\varepsilon < \frac{|\lambda|^{1/\theta}}{2}$. Now, since $\|x - x_n\|_r \to 0$ we have that

$$x = x_1 + \sum_{n=1}^\infty x_{n+1} - x_n$$

in L_r, and if $\varepsilon < \frac{|\lambda|^{1/\theta}}{2}$, this also holds in L_q. This means that for some constant $C_{q,r} > 0$ we have

$$\|x\|_r \le \|x\|_q \le C_{q,r}\|x\|_r$$

for every $x \in X_\lambda$. $\square$

We have seen that disjointly strictly singular operators have in some cases very nice spectral properties. However, despite DSS operators are closely related to strictly singular, the spectra of the former does not have any structure in general as the following shows.

Example 1. Given any compact set $K \subset \mathbb{C}$, there exists a DSS operator $T : L_p \to L_p$ with $1 < p < \infty$ ($p \ne 2$), such that $\sigma(T) = K \cup \{0\}$.

Proof. Indeed, given a compact set $K \subset \mathbb{C}$, let $\{\lambda_n\}_{n=1}^\infty$ be a dense sequence in K. Let $T : L_p \to L_p$ be defined by

$$\begin{tikzcd} L_p \arrow[r, "T"] \arrow[d, "P"'] & L_p \\ {[r_n]} \arrow[r, "m"] & {[r_n]} \arrow[u, "J"'] \end{tikzcd}$$

where (r_n) denotes the Rademacher functions which span a complemented subspace in L_p, P is the corresponding projection, J is an isomorphic embedding, and $m : [r_n] \to [r_n]$ is defined by

$$m\left(\sum_{n=1}^{\infty} a_n r_n\right) = \sum_{n=1}^{\infty} a_n \lambda_n r_n.$$

It is clear that $T = JmP$ is a *DSS* operator, since so is P (notice that every sequence of disjoint elements in $L_p[0, 1]$ is equivalent to the unit vector basis of ℓ_p while the sequence of Rademacher functions (r_n) is equivalent to the unit vector basis of ℓ_2). Moreover, λ_n is an eigenvalue of T for every n, so in particular $K = \overline{\{\lambda_n\}_{n=1}^{\infty}} \subset \sigma(T)$. Since $0 \in \sigma(T)$ always holds, we have the inclusion $K \cup \{0\} \subset \sigma(T)$.

For the converse, let $\lambda \notin K$, $\lambda \neq 0$ and pick $\delta > 0$ such that $|\lambda - \lambda_n| > \delta$ for every n. This allows us to consider the operator $S_\lambda : L_p \to L_p$ as follows. Let $L_p = [r_n] \oplus Y$, and define

$$\begin{array}{cccc} S_\lambda : & [r_n] \oplus Y & \longrightarrow & [r_n] \oplus Y \\ & \displaystyle\sum_{n=1}^{\infty} a_n r_n + y & \longmapsto & \displaystyle\sum_{n=1}^{\infty} \frac{a_n}{\lambda - \lambda_n} r_n + \frac{1}{\lambda} y. \end{array}$$

Since $\delta > 0$ and $\lambda \neq 0$ it is clear that S_λ is bounded. A straightforward computation shows also that

$$(\lambda - T)S_\lambda = S_\lambda(\lambda - T) = I.$$

This proves that $\lambda \notin \sigma(T)$, so we have $\sigma(T) = K \cup \{0\}$. $\square$

The following example provides a positive DSS operator which is not strictly singular nor even Riesz.

Example 2. Let $\Delta = \{-1, 1\}^{\mathbb{N}}$ be the Cantor group endowed with its Haar measure $\mu = \Pi_{n=1}^{\infty} \mu_n$, where $\mu_n(-1) = \mu_n(1) = \frac{1}{2}$. For a fixed sequence $(\varepsilon_n)_n$ in $(0, 1)$ converging to some $\varepsilon \in (0, 1)$ with $\sup_n \varepsilon_n < 1$, let us consider $\nu = \Pi_{n=1}^{\infty} \nu_{\varepsilon_n}$, where $\nu_{\varepsilon_n}(1) = \frac{1+\varepsilon_n}{2}$ and $\nu_{\varepsilon_n}(-1) = \frac{1-\varepsilon_n}{2}$. Let

$$(Tf)(x) = \int_\Delta f(xy)d\nu(y).$$

T is a positive DSS operator on $L_p(\Delta)$ for $1 < p < 2$ whose point spectrum contains the set $\{\varepsilon_{n_1} \cdot \ldots \cdot \varepsilon_{n_k} : n_1 < \cdots < n_k,\ k \in \mathbb{N}\}$.

Proof. Since T is defined as convolution by the probability measure ν, it is a contraction on $L_p(\Delta)$ for every $1 \le p \le \infty$. Indeed,

$$\begin{aligned} \|Tf\|_p &= \left(\int_\Delta \left|\int_\Delta f(xy)d\nu(y)\right|^p d\mu(x)\right)^{\frac{1}{p}} \\ &\le \int_\Delta \left(\int_\Delta |f(xy)|^p d\mu(x)\right)^{\frac{1}{p}} d\nu(y) \\ &= \int_\Delta \|f\|_p d\nu(y) \\ &= \|f\|_p. \end{aligned}$$

Let us consider the characters on Δ given by $r_n(x) = x_n$. It is clear that

$$(Tr_n)(x) = \int_\Delta r_n(xy)d\nu(y) = r_n(x)\int_\Delta r_n(y)d\nu(y) = \varepsilon_n r_n(x).$$

And similarly, for any finite set $A = \{n_1, \ldots, n_k\} \subset \mathbb{N}$ if we denote $w_A = r_{n_1} \cdot \cdots \cdot r_{n_k}$, we get

$$Tw_A = \varepsilon_{n_1} \cdot \ldots \cdot \varepsilon_{n_k} w_A.$$

This shows the last assertion of the claim concerning the point spectrum of T.

We claim that T is in fact bounded from $L_p(\Delta)$ to some $L_r(\Delta)$ with $p < r$. To show this, by interpolation, it is enough to prove that for some $s < 2$, $T : L_s(\Delta) \to L_2(\Delta)$ is bounded.

It is well known that the family $\{w_A : A \subset \mathbb{N},\ |A| < \infty\}$ forms an orthogonal basis of $L_2(\Delta)$ (called the Walsh basis). Let W_n denote the linear span of $\{w_A : |A| = n\}$, and notice that the union $\bigcup_n W_n$ is dense in $L_p(\Delta)$, for any $1 \le p < \infty$. Moreover, for $s < 2$ there is a constant C_s (which tends to 1 as $s \to 2$) so that for all $f \in W_n$,

$$\|f\|_2 \le C_s^n \|f\|_s$$

(see [23, Section 5], [25]). Let $1 < s < 2$ be such that $\sup_j \varepsilon_j C_s < 1$. Now, for $f \in W_n$, using the orthogonality of w_A we have

$$\|Tf\|_2 \leq (\sup_j \varepsilon_j)^n \|f\|_2 \leq (C_s \sup_j \varepsilon_j)^n \|f\|_s.$$

Therefore, since $\sup_j \varepsilon_j C_s < 1$, for any $f \in \bigcup_n W_n$ we have $\|Tf\|_2 \leq \|f\|_s$. Hence, by the density of $\bigcup_n W_n$ in L_s, we see that $T : L_s(\Delta) \to L_2(\Delta)$ is bounded for some $s < 2$, as desired.

This proves that $T : L_p(\Delta) \to L_p(\Delta)$ is disjointly strictly singular, since it factors through $L_r(\Delta)$ for some $r > p$, and ℓ_p is not isomorphic to any subspace of $L_r(\Delta)$. □

Notice, that for every k, ε^k is an accumulation point in the spectrum of the above defined operator. Hence, this operator is not a Riesz operator.

4. Complementedly strictly singular operators

In this section we introduce a new class of operators related to strictly singular operators. We will study their relation with disjointly strictly singular operators as well as their spectral properties.

Definition 1. Given Banach spaces X and Y, an operator $T : X \to Y$ is called complementedly strictly singular (CSS) if for any complemented subspace $Z \subset X$ such that the restriction $T|_Z$ is an invertible operator we must have $\dim(Z) < \infty$.

We present now some basic properties of the class of CSS operators:

Proposition 4. *$CSS(X, Y)$ is closed in $L(X, Y)$.*

Proof. Let $T_n \in CSS(X, Y)$ be such that $\|T_n - T\| \to 0$ for some $T \in L(X, Y)$. Suppose $T \notin CSS(X, Y)$, then there exists a complemented subspace $M \subset X$ with infinite dimension, such that the restriction $T|_M$ is invertible. Therefore, for some $\alpha > 0$ and every $x \in M$ we have $\|Tx\| \geq \alpha\|x\|$.

Let $n_0 \in \mathbb{N}$ be such that $\|T - T_{n_0}\| \leq \frac{\alpha}{2}$. Thus, for each $x \in M$ we have

$$\|T_{n_0}x\| \geq \|Tx\| - \|(T - T_{n_0})x\| \geq \alpha\|x\| - \frac{\alpha}{2}\|x\| = \frac{\alpha}{2}\|x\|.$$

This means that T_{n_0} is invertible on M, and this is a contradiction with the fact that T_{n_0} is CSS. □

Clearly, every strictly singular operator is a CSS operator, however the converse is not true.

Example 3. A CSS operator which is not strictly singular.

Proof. We use the construction given in [15, Theorem C]. Recall that $L_r(\ell_q)$ denotes the Banach lattice which consists of sequences $x = (x_1, x_2, \ldots)$ of elements in L_r such that

$$\|x\|_{L_r(\ell_q)} = \left\| \left(\sum_{n=1}^{\infty} |x_n|^q \right)^{\frac{1}{q}} \right\|_{L_r} < \infty.$$

Given $1 < r < p < 2 < q < \infty$ and $s \in (p, 2)$ there exists an operator $T : L_p \to L_r(\ell_q)$ which is not invertible on any subspace of L_p isomorphic to ℓ_2 nor ℓ_p, but it is invertible on a subspace isomorphic to ℓ_s.

Clearly this operator is not strictly singular. Yet, since every complemented subspace of L_p is either isomorphic to ℓ_2 or contains a complemented subspace isomorphic to ℓ_p [21], it follows that T is CSS. □

However, there is a family of spaces where the class of CSS operators coincides with that of strictly singular. Recall that a Banach space X is called subprojective if every infinite dimensional subspace $M \subset X$, contains another subspace $N \subset M$ which is also infinite dimensional and complemented in X. Hence, it is clear that if X is subprojective every operator $T : X \to Y$ is strictly singular if and only if it is complementedly strictly singular. The family of subprojective spaces includes the spaces ℓ_p $(1 \leq p < \infty)$, c_0, $L_p(\mu)$ for $p \geq 2$, and several other examples (see [26]).

It is worth noting that a compact perturbation of a CSS operator is also CSS:

Proposition 5. *Let $T : X \to Y$ be a CSS operator. If $S : X \to Y$ is compact, then $T + S$ is also CSS.*

Proof. Let us suppose that $(T + S)|_M$ is invertible for some $M \subset X$ with $\dim(M) = \infty$. Thus, there is $\alpha > 0$ such that $\|(T + S)x\| \geq \alpha\|x\|$ for every $x \in M$. Since S is compact, there exists $N \subset M$ of finite codimension in M with $\|S|_N\| < \frac{\alpha}{2}$ (cf. [27, III.2.3]). Therefore, for every $x \in N$ we have

$$\|Tx\| \geq \|(T + S)x\| - \|Sx\| \geq \alpha\|x\| - \frac{\alpha}{2}\|x\| = \frac{\alpha}{2}\|x\|.$$

Hence, T is invertible on N but since T is CSS, N cannot be complemented in X. Moreover, since $\dim(M/N) < \infty$, the subspace M cannot be complemented in X either. □

In connection with this result, a natural question arises: Is the class of CSS operators between two Banach spaces a linear subspace of the bounded operators?

4.1. CSS vs. DSS

It is well-known that every sequence of disjoint functions on L_p $(1 \leq p < \infty)$ spans a complemented subspace isomorphic to ℓ_p. It follows that every CSS operator $T : L_p \to Y$ is necessarily DSS. This fact can be extended to the class of disjointly subprojective Banach lattices. Recall that a Banach lattice E is called disjointly subprojective if for every disjoint sequence (f_n) in E, there is a sequence (g_n) of blocks of (f_n), such that their span $[g_n]$ is complemented in E. The family of disjointly subprojective Banach lattices includes L_p spaces, Lorentz $L_{p,q}$ and Λ_w^p spaces (for $1 \leq p < \infty$) [28].

Although, in general, the classes of DSS and CSS operators need not coincide, on some spaces they do.

Proposition 6. *For any Banach space Y, every operator $T : L_1(\mu) \to Y$ is CSS if and only if it is DSS.*

Proof. As mentioned above, since every disjoint sequence in $L_1(\mu)$ spans a complemented subspace isomorphic to ℓ_1, if T is CSS, then it must be DSS. Conversely, suppose T is DSS but there is an infinite dimensional subspace $X \subset L_1(\mu)$ such that $T|_X$ is invertible. We claim that this subspace must be reflexive and hence cannot be complemented. Indeed, if X contains a sequence equivalent to the unit vector basis of ℓ_1, then by Dor [29] T would be invertible on the span of a disjoint sequence equivalent to the unit vector basis of ℓ_1. Since T is DSS this cannot happen, so X does not contain any subspace isomorphic to ℓ_1. It follows that X must be reflexive (cf. [9, Vol. II, Theorem 1.c.5]). □

In general, the classes of CSS and DSS operators are incomparable. The simplest example of a DSS operator which is not CSS is given by the projection on the span of the Rademacher functions on L_p for any $p \neq 2$, $P : L_p \to L_p$. Indeed, this operator is clearly non CSS since it is invertible on the span of the Rademacher functions, but it is DSS since every disjoint sequence in L_p spans a subspace isomorphic to ℓ_p.

The next example requires a bit more technology.

Example 4. A CSS operator which is not DSS.

Proof. We build this example by a simple modification of [15, Theorem C]. Let $T : L_p \to L_r(\ell_q)$ be the operator given by this result. Consider now the space H_p which is linearly isomorphic to L_p and is a discrete Banach lattice with the order induced by the unconditional Haar basis. Let $H : H_p \to L_p$ denote the corresponding isomorphism, and consider the operator

$$TH : H_p \to L_r(\ell_q).$$

As in Example 3, the operator TH is CSS since it is not invertible on any subspace isomorphic to ℓ_p nor ℓ_2 and every complemented subspace of H_p (which is isomorphic to L_p) must contain one of these spaces. However, the operator TH is not DSS, since by construction, the operator T is invertible on the span of a sequence (g_n) equivalent to the unit vector basis of ℓ_s (with $p < s < 2$). Using a perturbation argument [9, Vol. I, Prop. 1.a.11] it is easy to see that one can take a block sequence of the Haar basis in L_p arbitrarily close to (g_n) so that TH is invertible on this disjoint sequence in H_p spanning ℓ_s. □

4.2. Spectra of CSS operators

Let us discuss now the spectral properties of CSS operators. Clearly, if X is infinite dimensional, then 0 is in the spectrum of any CSS operator $T : X \to X$.

Given an operator $T : X \to X$, recall that a subset $\sigma \subset \sigma(T)$ is called a spectral set of T if both σ and $\sigma(T) \setminus \sigma$ are closed in the relative topology of $\sigma(T)$. It follows from the well-known Spectral Mapping Theorem (cf. [8, Section 6.4]) that to any non-trivial spectral set σ of an operator T we can associate two complemented subspaces Y_σ, Z_σ of X such that $X = Y_\sigma \oplus Z_\sigma$ with $T(Y_\sigma) \subset Y_\sigma$, $T(Z_\sigma) \subset Z_\sigma$ in such a way that $\sigma(T|_{Y_\sigma}) = \sigma$ and $\sigma(T|_{Z_\sigma}) = \sigma(T) \setminus \sigma$.

Lemma 2. *Let X be a Banach space and $T : X \to X$ a CSS operator. Any non-trivial spectral set $\sigma \subset \sigma(T)$ with $0 \notin \sigma$ is finite.*

Proof. Let σ be a non-trivial spectral set such that $0 \notin \sigma$. Hence, as was mentioned above there exist complemented subspaces Y_σ, Z_σ of X with $T(Y_\sigma) \subset Y_\sigma$, $T(Z_\sigma) \subset Z_\sigma$ and $X = Y_\sigma \oplus Z_\sigma$, in such a way that

$$\sigma(T|_{Y_\sigma}) = \sigma \quad \text{and} \quad \sigma(T|_{Z_\sigma}) = \sigma(T) \setminus \sigma.$$

Since $0 \notin \sigma(T|_{Y_\sigma})$, it follows that $T|_{Y_\sigma}$ is invertible. However, T is a CSS operator, so we must have that $\dim(Y_\sigma) < \infty$. This implies that $\sigma = \sigma(T|_{Y_\sigma})$ is a finite set. □

Corollary 1. *The spectrum $\sigma(T)$ is a finite set if and only if 0 is an isolated point of $\sigma(T)$.*

Proof. Clearly if 0 is not isolated, then $\sigma(T)$ contains infinitely many points. For the converse, suppose that 0 is an isolated point in $\sigma(T)$. Then $\sigma(T) \setminus \{0\}$ is a non-trivial spectral set, so by Lemma 2 it is finite. It follows that $\sigma(T)$ is finite as well. □

Corollary 2. *All the accumulation points of $\sigma(T)$ belong to the connected component of $\sigma(T)$ containing $\{0\}$.*

Proof. Let $\lambda \in \sigma(T)$ be an accumulation point which is not in the connected component of $\sigma(T)$ containing $\{0\}$. Therefore, there exists two closed and open sets σ_1, σ_2 in $\sigma(T)$ with $\sigma_1 \cup \sigma_2 = \sigma(T)$ and such that $\lambda \in \sigma_1$ and $0 \in \sigma_2$. Now, it follows that σ_1 is a non-trivial spectral set with $0 \notin \sigma_1$, so by Lemma 2, σ_1 must be finite. However, since λ is an accumulation point of $\sigma(T)$ belonging to σ_1, and since σ_1 is open in $\sigma(T)$ it follows that σ_1 is not finite. This contradiction proves the result. □

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References

[1] P. Meyer-Nieberg, Banach Lattices, Springer-Verlag, 1991.
[2] H.H. Schaefer, Banach lattices and positive operators, in: Die Grundlehren der mathematischen Wissenschaften, Band 215, Springer-Verlag, 1974.
[3] Y.A. Abramovich, C.D. Aliprantis, Positive operators, in: Handbook of the Geometry of Banach Spaces, Vol. I, North-Holland, Amsterdam, 2001, pp. 85–122.
[4] R. Bürger, The mathematical theory of selection, recombination, and mutation, in: Wiley Series in Mathematical and Computational Biology, John Wiley & Sons, Ltd., Chichester, 2000.
[5] R. Bürger, Mutation-selection balance and continuum-of-alleles models, Math. Biosci. 91 (1988) 67–83.
[6] C.D. Aliprantis, O. Burkinshaw, Positive compact operators on Banach lattices, Math. Z. 174 (1980) 289–298.
[7] F.L. Hernández, B. Rodríguez-Salinas, On ℓ_p complemented copies in Orlicz spaces II, Israel J. Math. 68 (1989) 27–55.
[8] Y.A. Abramovich, C.D. Aliprantis, An invitation to operator theory, in: Graduate Studies in Mathematics, vol. 50, American Mathematical Society, Providence, RI, 2002.
[9] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces I and II, Springer-Verlag, Berlin-New York, 1979.
[10] T. Kato, Perturbation theory for nullity deficiency and other quantities of linear operators, J. Anal. Math. 6 (1958) 273–322.
[11] D. Dosev, W.B. Johnson, Commutators on ℓ_∞, Bull. Lond. Math. Soc. 42 (1) (2010) 155–169.
[12] W.T. Gowers, B. Maurey, The unconditional basic sequence problem, J. Amer. Math. Soc. 6 (4) (1993) 851–874.
[13] J. Flores, F.L. Hernández, P. Tradacete, Domination problems for strictly singular operators and other related classes, Positivity 16 (2011) 595–616.
[14] J. Flores, F.L. Hernández, P. Tradacete, Powers of operators dominated by strictly singular operators, Q. J. Math. 59 (2008) 321–334.
[15] J. Flores, F.L. Hernández, N.J. Kalton, P. Tradacete, Characerizations of strictly singular operators on Banach lattices, J. Lond. Math. Soc. 79 (2009) 612–630.
[16] J. Flores, P. Tradacete, V.G. Troitsky, Disjointly homogeneous Banach lattices and compact products of operators, J. Math. Anal. Appl. 354 (2) (2009) 657–663.
[17] P. Tradacete, Subspace structure of Lorentz $L_{p,q}$ spaces and strictly singular operators, J. Math. Anal. Appl. 367 (2010) 98–107.
[18] A. Pełczynśki, On strictly singular and strictly cosingular operators. I. Strictly singular and strictly cosingular operators in C(S)-spaces, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 13 (1965) 31–36.
[19] J. Bourgain, A characterization of non-Dunford-Pettis operators on L^1, Israel J. Math. 37 (1–2) (1980) 48–53.
[20] L. Weis, Integral operators and changes of density, Indiana Univ. Math. J. 31 (1) (1982) 83–96.
[21] M.I. Kadeč, A. Pełczyński, Bases, lacunary sequences and complemented subspaces in the spaces L_p, Studia Math. 21 (1962) 161–176.
[22] W.B. Johnson, L. Jones, Every L_p operator is an L_2 operator, Proc. Amer. Math. Soc. 72 (2) (1978) 309–312.
[23] W.B. Johnson, G. Schechtman, Multiplication operators on $L(L_p)$ and ℓ_p-strictly singular operators, J. Eur. Math. Soc. (JEMS) 10 (2008) 1105–1119.
[24] F.L. Hernández, E.M. Semenov, P. Tradacete, Strictly singular operators on L_p spaces and interpolation, Proc. Amer. Math. Soc. 138 (2010) 675–686.
[25] H.P. Rosenthal, Convolution by a biased coin, The Altgeld Book, University of Illinois, Functional Analysis Seminar, 1975/76.
[26] R.J. Whitley, Strictly singular operators and their conjugates, Trans. Amer. Math. Soc. 113 (1964) 252–261.
[27] S. Goldberg, Unbounded Linear Operators: Theory and Applications, McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966.
[28] N.L. Carothers, S.J. Dilworth, Geometry of Lorentz spaces via interpolation, Texas Functional Analysis Seminar 1985–1986, 107–133, Longhorn Notes, Univ. Texas, Austin, TX, 1986.
[29] L. Dor, On projections in L_1, Ann. of Math. 102 (1975) 463–474.